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1999 J. Phys. A: Math. Gen. 32 6219

(http://iopscience.iop.org/0305-4470/32/34/308)

J. Phys. A: Math. Gen. **32** (1999) 6219–6231. Printed in the UK PII: S0305-4470(99)00802-1

Inverse scattering transform on a finite interval

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Received 8 January 1999

Abstract. A procedure with an *effective S*-matrix is developed to apply the inverse scattering transform method to a finite interval. Second harmonic generation and stimulated Raman scattering are treated as two examples to demonstrate the applicability of this method.

1. Introduction

An important class of nonlinear integrable problems are those which have both an initial value and also a boundary value. Of these, the simplest ones are those of hyperbolic form, such as the sine–Gordon equation [1] and others of that structure [2]. For these equations, the initial value–boundary value problem is a well stated problem, and the existence and uniqueness of the solution is well established. These equations are also integrable. Thus one can also analyse these problems and study their solution by use of an inverse scattering transform (IST) on a finite or semi-infinite interval. Such an IST requires the solution of a direct and inverse scattering problem on the corresponding interval. This problem has several subtleties, which we will discuss here.

Earlier works include [3–6] which have described various formal aspects of this problem. Other papers [7–9] have treated this class of problems, with various special boundary values. Rather artificial types of initial–boundary value problems were treated in [10].

As first pointed out in [11–14], and as later detailed in [15, 16], the evolution of the scattering data along a boundary where there is boundary data, can differ strongly from what we are familiar with on the infinite interval, where the scattering data evolves only according to $\exp[i\Omega(\zeta)t]$. Even solitons can be created at such a boundary. Of course, such solutions can always be embedded in the infinite interval, and when this is done and looked at from that point of view, one can then understand the nature of the finite interval solution: it is simply a finite slice of the infinite interval and the more complex evolution of the scattering data is then simply the consequence of taking the solution to be zero outside of the finite interval.

One of the key problems in the finite interval case, is determining the evolution of the scattering data. Unfortunately, when one has a finite interval, one cannot resort to the argument that one expects the solution to vanish at infinity. Instead one must know the solution at the other end, in order to evaluate the evolution of the scattering data. However, this then overdetermines the problem. The first solution to this problem was outlined in connection with the IST solution of the stimulated Raman scattering (SRS) equations and the two-photon propagation (TPP)

0305-4470/99/346219+13$30.00

equations [11]. There, it was reasoned that for the IST in a finite interval, the respective boundary values at the right-hand end of the considered interval had to be irrelevant, and in fact, due to the nature of the characteristics, could be completely ignored. As an application, one of these authors (DJK) then later used this technique to obtain the full and complete IST solution for the low molecular excitation SRS equations [17]. He also demonstrated that the asymptotic form of the solution would break up into a transient part and a Painlevé III part, just as was typically seen in numerical simulations [18]. More recently, Fokas and Menyuk [19] have treated the same problem from a Riemann–Hilbert point of view, where they have compared the asymptotic form to the structures seen in the numerical solutions.

Here, we will develop the above ideas more rigorously and in more detail than has been done previously. In particular, we want to look at some of the practicalities involved in using this method for reconstructing solutions for finite (not asymptotical) times. For simplicity, this shall be done only for the AKNS systems. Also we shall give several applications of these results.

In section 2, the concept of the *effective* S-matrix is introduced and its equivalence—with respect to the considered finite interval—to the full S-matrix is proven. In section 3, we demonstrate by using a trivially solvable (‘C-integrable’) problem, that our procedure does yield the correct results. A typical initial-boundary, or Goursat problem, for stimulated Raman scattering (in the low excitation limit) is treated in section 4.

2. The effective S-matrix

Let us consider a general scattering problem of the AKNS type [20] and use the notation of the quoted paper,

$$\partial_x v = \begin{pmatrix} -\mathrm{i}\zeta & q(x) \\ r(x) & \mathrm{i}\zeta \end{pmatrix} v \equiv U v \tag{1}$$

where v means the column $v = (v_1, v_2)^T$. Now, however, we consider a finite interval $0 < x < x_f$ and define the Jost functions to be those solutions to (1) fulfilling

$$\phi(x=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \bar{\phi}(x=0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{2}$$

$$\psi(x=x_f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\zeta x_f} \qquad \bar{\psi}(x=x_f) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\zeta x_f}. \tag{3}$$

The scattering coefficients, $a, b, \bar{a}, \bar{b}$, are then determined by

$$\begin{aligned} \phi(x) &= a\bar{\psi}(x) + b\psi(x) \\ \bar{\phi}(x) &= \bar{b}\bar{\psi}(x) - \bar{a}\psi(x). \end{aligned} \tag{4}$$

These definitions coincide exactly with those given in [20] for the *infinite* interval, providing the potentials q, r are extended to the full x-axis by setting them equal to zero outside of the finite interval $0 < x < x_f$.

For any solution, $v(x)$, to (1), it holds that

$$v(x_f) = \mathrm{e}^{-\mathrm{i}\sigma_3\zeta x_f} \boldsymbol{S} v(0) \tag{5}$$

where $\boldsymbol{S}$ is the scattering matrix

$$\boldsymbol{S} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}. \tag{6}$$

Now, the key feature of the AKNS scheme, is that for any integrable system that is solvable by the AKNS IST, there will exist a common solution, $v(x,t)$, of (1) and the evolution equations

$$\partial_t v = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v \equiv Vv \tag{7}$$

where A, B, C are some rational functions of ζ. In fact, for the hyperbolic systems of interest here, it is well known that all these coefficients will vanish for $|\zeta| \to \infty$. This is a key feature that we shall use later.

By differentiation of (5) with respect to t, it follows that

$$\partial_t \boldsymbol{S} = -\boldsymbol{S}V_i + EV_f E^{-1}\boldsymbol{S} \tag{8}$$

where $V_i = V(0,t)$, $V_f = V(x_f,t)$ and

$$E = \begin{pmatrix} \exp(\mathrm{i}\zeta x_f) & 0 \\ 0 & \exp(-\mathrm{i}\zeta x_f) \end{pmatrix}. \tag{9}$$

As is clear from (8), one must know $V(x,t)$ for $x = 0$ *and* $x = x_f$, if one is to use this relation, to determine the time evolution of the scattering *coefficients*. However, within the inverse scattering procedure, we do not need the full set of scattering coefficients, $a, b, \bar{a}, \bar{b}$. Rather, we only need a subset of this information, and for no other purpose but to compute the two functions

$$G(z) = \frac{1}{2\pi}\int_{\mathcal{C}} c(\zeta)\,\mathrm{e}^{-\mathrm{i}\zeta z}\,\mathrm{d}\zeta \qquad z < 2x_f \tag{10}$$

$$\bar{G}(z) = \frac{1}{2\pi}\int_{\bar{\mathcal{C}}} \bar{c}(\zeta)\,\mathrm{e}^{\mathrm{i}\zeta z}\,\mathrm{d}\zeta \qquad z < 2x_f \tag{11}$$

where the *reflection coefficients*, c and $\bar{c}$, are defined as

$$c = \frac{\bar{b}}{a} \qquad \bar{c} = \frac{b}{\bar{a}} \tag{12}$$

and the contour $\mathcal{C}$ ($\bar{\mathcal{C}}$) of integration goes from $-\infty$ to $+\infty$ in the complex ζ-plane passing over (under) all zeros of a ($\bar{a}$).

Now, from the theory of characteristics, it follows that the solution for V_f is fully dependent on V_i and the initial data. Thus we first seek to find a way to factor the V_f-dependence out of the S-matrix, leaving a V_f independent part.

This can be done as follows. Write the S-matrix as the product

$$\boldsymbol{S} = \boldsymbol{S}_1\boldsymbol{S}^{\mathrm{eff}} \tag{13}$$

with its factors being determined by

$$\partial_t \boldsymbol{S}^{\mathrm{eff}} = -\boldsymbol{S}^{\mathrm{eff}}V_i \qquad \boldsymbol{S}^{\mathrm{eff}}(0) = \boldsymbol{S}(0) \tag{14}$$

$$\partial_t \boldsymbol{S}_1 = EV_f E^{-1}\boldsymbol{S}_1 \qquad \boldsymbol{S}_1(0) = \mathbf{1}. \tag{15}$$

Introduce a third matrix, $\boldsymbol{S}_2$, which satisfies

$$\partial_t \boldsymbol{S}_2 = V_f\boldsymbol{S}_2 \qquad \boldsymbol{S}_2(0) = \mathbf{1} \tag{16}$$

then it follows that

$$\boldsymbol{S}_1 = E\boldsymbol{S}_2E^{-1} \tag{17}$$

is the solution of (15). We will show now that the S-matrix, $\boldsymbol{S}$, can be replaced by $\boldsymbol{S}^{\text{eff}}$, or in other words, the term containing V_f in (8) does not affect the result.

Theorem. *The effective S-matrix, $\boldsymbol{S}^{\text{eff}}(t)$, defined by (8), gives rise to the same functions, $G(z)$, $\bar{G}(z)$, on the interval $0 < z < 2x_f$, as does the complete S-matrix, $\boldsymbol{S}$. Thus the Marchenko equations in both cases lead to the same potentials within the physical interval $0 < x < x_f$.*

Proof. First, we will establish bounds on the asymptotic form of any scattering coefficients (as functions of ζ), for any potential on a finite interval. As in [20], from (1)–(3), it is quite straightforward to show, for ζ in the upper-half complex ζ-plane, that

$$|\bar{a}(\zeta)| \leqslant 1 + [\cosh(Q) - 1]\,\mathrm{e}^{-2\eta x_f} \tag{18}$$

$$|b(\zeta)| \leqslant \sinh(Q)\,\mathrm{e}^{-2\eta x_f} \tag{19}$$

where $\eta = \operatorname{Im}\zeta$ and $Q = \int_0^{x_f} \max(|q(x)|, |r(x)|)\,\mathrm{d}x$. Next, we want to establish asymptotic limits on the matrix $\boldsymbol{S}_2$. For this purpose, we define the components of this matrix in the following way:

$$\boldsymbol{S}_2 \equiv \begin{pmatrix} a_2 & -\bar{b}_2 \\ b_2 & \bar{a}_2 \end{pmatrix} \tag{20}$$

and define c_2 and $\bar{c}_2$ analogously to (12). According to our previous remarks, we take the matrix elements of V_f to decay, at least algebraically in ζ, for $|\zeta| \to \infty$. We will assume that $V_f \to \mathrm{O}(1/\zeta)$ in this limit. It now follows from (16) that the solution for $\boldsymbol{S}_2$ will algebraically approach $\mathbf{1} + \mathrm{O}(1/\zeta)$ as $|\zeta| \to \infty$. Furthermore, it follows for $|\zeta|$ sufficiently large, c_2 will not only be analytic, but also will vanish as $\mathrm{O}(1/\zeta)$, for $|\zeta| \to \infty$ in the upper-half complex ζ-plane.

Once we have these limits on the scattering coefficients and on $\boldsymbol{S}_2$, then we can proceed and construct $\boldsymbol{S}_1$,

$$\boldsymbol{S}_1 \equiv \begin{pmatrix} a_2 & -\bar{b}_2\mathrm{e}^{2\mathrm{i}\zeta x_f} \\ b_2\mathrm{e}^{-2\mathrm{i}\zeta x_f} & \bar{a}_2 \end{pmatrix}. \tag{21}$$

Next, we take the matrix $\boldsymbol{S}^{\text{eff}}$, and define its coefficients by

$$\boldsymbol{S}^{\text{eff}} \equiv \begin{pmatrix} a^{\text{eff}} & -\bar{b}^{\text{eff}} \\ b^{\text{eff}} & \bar{a}^{\text{eff}} \end{pmatrix}. \tag{22}$$

Note that in (14), in the evolution of these coefficients, only $\bar{a}^{\text{eff}}$ and b^{eff} are coupled (and also only a^{eff} and $\bar{b}^{\text{eff}}$ are coupled). Thus along the contour $\mathcal{C}$, for $|\zeta|$ sufficiently large, the evolution of $\bar{a}^{\text{eff}}$ and b^{eff} in t will still obey the same above estimates for $\bar{a}$ and b, if $Q(x)$ is replaced by its evolution in t.

Now define c^{eff} and $\bar{c}^{\text{eff}}$ as in (12). Then from (10), (11) and (13), we obtain

$$c = \frac{c^{\text{eff}} + (\bar{a}^{\text{eff}}/a^{\text{eff}})c_2\mathrm{e}^{2\mathrm{i}\zeta x_f}}{1 - (b^{\text{eff}}/a^{\text{eff}})c_2\mathrm{e}^{2\mathrm{i}\zeta x_f}}. \tag{23}$$

Consider (23) for ζ along the contour $\mathcal{C}$, with $|\zeta|$ large in the upper-half ζ-plane. We have

$$c = c^{\text{eff}} + \left[\bar{a}^{\text{eff}} + c^{\text{eff}}b^{\text{eff}}\right]\frac{c_2}{a^{\text{eff}}}\mathrm{e}^{2\mathrm{i}\zeta x_f}\sum_{\nu=0}^{\infty}\left[\frac{b^{\text{eff}}}{a^{\text{eff}}}c_2\mathrm{e}^{2\mathrm{i}\zeta x_f}\right]^{\nu}. \tag{24}$$

Obviously, as long as $0 < z < 2x_f$, the integrand $c\mathrm{e}^{-\mathrm{i}\zeta z}$ in (10) differs from $c^{\text{eff}}\mathrm{e}^{-\mathrm{i}\zeta z}$ only by terms which are decaying exponentially in the upper-half ζ-plane and therefore do not contribute to the integral in (10).

In quite an analogous way, it follows that in (11) $\bar{c}$ can be replaced by $\bar{c}^{\text{eff}}$, and the proof is complete. $\square$

Note, however, that $\boldsymbol{S}^{\text{eff}}$ then cannot be interpreted as a *scattering matrix* in the proper sense. The coefficients of $\boldsymbol{S}^{\text{eff}}$ correspond to the scattering coefficients of some potential, which is exactly $Q(x)$ on the interval $0 < x < x_f$, but could be very different outside of this interval.

From $\boldsymbol{S}^{\text{eff}}$ we find $c^{\text{eff}}, \bar{c}^{\text{eff}}(t, \zeta)$. Then we have to execute the integrals in (10) and (11) in order to obtain the functions $G^{\text{eff}}(z)$ and $\bar{G}^{\text{eff}}(z)$. Note that due to the analyticity of $c^{\text{eff}}(t, \zeta)$ and $\bar{c}^{\text{eff}}(t, \zeta)$ in the limit of $|\zeta| \to \infty$, the resulting $G^{\text{eff}}(z)$ and $\bar{G}^{\text{eff}}(z)$ both vanish for $z < 0$. Once we have these functions, we then turn to the Marchenko-type equations, cf [20],

$$\bar{L}(x, y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} G(x + y) - \int_{-\infty}^{x} L(x, s) G(s + y)\, \mathrm{d}s = 0 \qquad (x > y) \tag{25}$$

$$L(x, y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{G}(x + y) + \int_{-\infty}^{x} \bar{L}(x, s) \bar{G}(s + y)\, \mathrm{d}s = 0 \qquad (x > y) \tag{26}$$

where now we omit the superscript ‘eff’. Here L and $\bar{L}$ are two-component columns, $L \equiv (L_1, L_2)^T$, $\bar{L} \equiv (\bar{L}_1, \bar{L}_2)^T$.

The system may simplify by typical symmetry reductions, following [20].

(a) When $r = \epsilon q$, where $\epsilon = \pm 1$, we find

$$\bar{c}(\zeta) = -\epsilon c(-\zeta) \qquad \bar{G}(z) = -\epsilon G(z) \tag{27}$$

$$\bar{L}_1 = -\epsilon L_2 \qquad \bar{L}_2 = -L_1. \tag{28}$$

(b) When $r = \epsilon q^*$ where the star denotes complex conjugation we find

$$\bar{c}(\zeta) = -\epsilon c^*(\zeta^*) \qquad \bar{G}(z) = -\epsilon G^*(z) \tag{29}$$

$$\bar{L}_1 = -\epsilon L_2^* \qquad \bar{L}_2 = -L_1^*. \tag{30}$$

Eventually, from the solution of (25) and (26), the *potential* is found by

$$q(x) = 2\bar{L}_1(x, x) \qquad (0 < x < x_f). \tag{31}$$

3. Second harmonic generation

Let us now illustrate the above results with a simple example, which will demonstrate that this does work. We will use the second harmonic generation (SHG) equations. When they are reduced to pure amplitude modulation, we have (see, e.g., [21])

$$\partial_\chi q_1 = -2 q_2 q_1 \qquad \partial_\tau q_2 = q_1^2. \tag{32}$$

q_1 and q_2 are slowly varying amplitudes of the fundamental and harmonic waves, respectively, and χ, τ are characteristic coordinates in two-dimensional spacetime. The Goursat (or ‘initial–boundary’) problem

$$\begin{aligned} &q_1(\chi, 0) = 1 \qquad 0 < \chi < \chi_f \\ &q_2(0, \tau) = 0 \qquad 0 < \tau < \tau_f \end{aligned} \tag{33}$$

has the solution

$$q_1 = \frac{1}{1 + 2\chi\tau} \qquad q_2 = \frac{\tau}{1 + 2\chi\tau} \tag{34}$$

which can be easily checked by direct inspection. Now let us turn to the inverse scattering technique, and see whether we can rederive this solution directly.

The above system (32) fits into the scheme of section 2 upon taking the coordinates (x, t) to be (χ, τ) instead, and putting

$$U = \begin{pmatrix} -\mathrm{i}\zeta & 2q_2 \\ 2q_2 & \mathrm{i}\zeta \end{pmatrix} \tag{35}$$

$$V = \frac{\mathrm{i}}{\zeta} q_1^2 M \qquad M \equiv \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{36}$$

Due to (33), the initial S-matrix is trivial.

$$\boldsymbol{S}(\tau = 0) = \mathbf{1}. \tag{37}$$

Meanwhile, the τ-evolution of the *effective* S-matrix is determined by

$$\partial_\tau \boldsymbol{S}^{\mathrm{eff}} = -\boldsymbol{S}^{\mathrm{eff}} V_i \qquad \boldsymbol{S}^{\mathrm{eff}}(0) = \boldsymbol{S}(0) = \mathbf{1}. \tag{38}$$

This is easily solved and the solution is given by

$$\boldsymbol{S}^{\mathrm{eff}}(\tau) = \mathbf{1} - \frac{\mathrm{i}\tau}{\zeta} M. \tag{39}$$

The *effective* reflection coefficient then is found to be

$$c(\tau, \zeta) = -\frac{S_{12}}{S_{11}} = \frac{-\mathrm{i}\tau}{\zeta + \mathrm{i}\tau} \tag{40}$$

where the superscript 'eff' has been omitted. Note that c has a pole at $\zeta = \zeta_1 = -\mathrm{i}\tau$, which is in the lower-half ζ-plane. Furthermore, c is analytic in ζ everywhere else, including $\zeta \to \infty$. To construct G and $\bar{G}$, we take the Fourier transform, and find

$$G(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\zeta)\, \mathrm{e}^{-\mathrm{i}\zeta z}\, \mathrm{d}\zeta = -\theta(z)\, \tau\, \mathrm{e}^{-\tau z} \tag{41}$$

where $\theta(z)$ denotes the Heaviside step function. We recall that $G(z)$, for $z < 2\chi_f$, does not depend on whether it is taken from the effective reflection coefficient or from the proper one.

Now we wish to solve the inverse scattering problem in the region $0 < \chi < \chi_f$, and for some arbitrarily fixed value of τ. From the Marchenko-type equations, (25) and (26), we obtain the closed system

$$\bar{L}_1(\chi, y) + G(\chi + y) - \int_{-\infty}^{\chi} L_1(\chi, s) G(s + y)\, \mathrm{d}s = 0 \tag{42}$$

$$L_1(\chi, y) + \int_{-\infty}^{\chi} \bar{L}_1(\chi, s) \bar{G}(s + y)\, \mathrm{d}s = 0 \tag{43}$$

which is valid for $\chi > y$. In our case, we have $(r = q)$, and thus it holds that

$$\bar{G}(z) = -G(z). \tag{44}$$

The potential $2q_2$, will be found from the solution of (42) and (43) by

$$q_2(\chi) = \bar{L}_1(\chi, \chi). \tag{45}$$

It is easy to see that, since $G(z) = 0$ for $z < 0$, it follows that $q(\chi) = 0$ for $\chi < 0$.

Now, for $\chi + y > 0$, equations (42) and (43) reduce to

$$\bar{L}_1(\chi, y) + \tau e^{-\tau y} \int_{-y}^{\chi} L_1(\chi, s)\, e^{-\tau s}\, ds = \tau e^{-\tau(\chi+y)} \tag{46}$$

$$L_1(\chi, y) + \tau e^{-\tau y} \int_{-y}^{\chi} \bar{L}_1(\chi, s)\, e^{-\tau s}\, ds = 0. \tag{47}$$

Note that the integrands only require $L_1(\chi, y)$ and $\bar{L}_1(\chi, y)$ for $y < \chi$. The solutions for $y > \chi$ follow directly upon knowing the solutions in the region $-\chi < y < \chi$. So, let us restrict the above to $-\chi < y < \chi$, and then proceed to find the solutions in this region. Now we have

$$\bar{L}_1(\chi, y) + \tau \int_{-\chi}^{y} L_1(\chi, -s)\, e^{\tau(s-y)}\, ds = \tau e^{-\tau(\chi+y)} \tag{19$'$}$$

$$L_1(\chi, -y) + \tau \int_{y}^{\chi} \bar{L}_1(\chi, s)\, e^{\tau(y-s)}\, ds = 0. \tag{20$'$}$$

Equivalently, these integral equations can be replaced by the ordinary differential equations

$$\partial_y \begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} = M\tau \begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} \tag{48}$$

together with the conditions

$$\bar{L}_1(\chi, -\chi) = 1 \qquad L_1(\chi, -\chi) = 0. \tag{49}$$

Equation (48) has the solution

$$\begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} = (\mathbf{1} + (\chi + y)\tau M) \begin{pmatrix} \bar{L}_1(\chi, -\chi) \\ L_1(\chi, \chi) \end{pmatrix}. \tag{50}$$

From

$$\begin{pmatrix} \bar{L}_1(\chi, \chi) \\ L_1(\chi, -\chi) \end{pmatrix} = (\mathbf{1} + 2\chi\tau M) \begin{pmatrix} \bar{L}_1(\chi, -\chi) \\ L_1(\chi, \chi) \end{pmatrix} \tag{51}$$

and (49), we find that the solution for the potential is

$$q_2 \equiv \bar{L}_1(\chi, \chi) = \frac{1}{1 + 2\chi\tau} \tag{52}$$

$$L_1(\chi, \chi) = \frac{-2\chi\tau}{1 + 2\chi\tau} \tag{53}$$

in accordance with (34). Note that the result does not contain χ_f. This is due to the fact that we started from trivial initial conditions $q_2(\chi, 0) \equiv 0$. Obviously, this result can be taken to be the solution for the semi-line $0 < \chi < \infty$ as well.

This simple example has been used here to demonstrate the workability of the method. From the physical point of view, the Goursat problem for SHG is of restricted interest only. Instead a Cauchy problem has to be solved, which has been treated in some detail in [21], as well with the restriction to pure amplitude modulation.

4. Stimulated Raman scattering

Stimulated Raman scattering (SRS) (under the conditions of low excitation and neglect of damping) is described by the equations

$$\partial_\chi A_1 = -XA_2 \qquad \partial_\chi A_2 = X^* A_1 \qquad \partial_\tau X = A_1 A_2^*. \tag{54}$$

Now we restrict to real amplitudes A_1, A_2, X, and for convenience we assume

$$A_1^2 + A_2^2 = 1 \tag{55}$$

where the latter condition could always be achieved by proper (nonlinear) time scaling. Then clearly we could introduce an angle $u(\chi, \tau)$ with

$$A_1 = \cos(u/2) \qquad A_2 = \sin(u/2). \tag{56}$$

Together with (54), this implies $X = u_\chi/2$, and this leads to the sine–Gordon equation

$$u_{\chi\tau} = \sin u. \tag{57}$$

Let us apply these results to the experiment of Drühl *et al* [22]. In that experiment, it was assumed (with very good indication of such), that the soliton pulses observed were generated whenever there was a sign-flip in the Stokes field. So, let us consider such an initial–boundary value problem.

Let us take

$$\begin{aligned}
&A_1(0,\tau) = \cos(u_0/2) && 0 < \tau < \tau_f = \tau_1 + \tau_2 \\
&A_2(0,\tau) = \begin{cases} \sin(u_0/2) & 0 < \tau < \tau_1 \\ -\sin(u_0/2) & \tau_1 < \tau < \tau_f \end{cases} \\
&X(\chi, 0) = 0 && 0 < \chi < \chi_f
\end{aligned} \tag{58}$$

with $0 < u_0 = \text{constant} \leqslant \pi/2$. The physical interpretation of these boundary–initial value conditions is the following. From the boundary values, we see that at the front of the nonlinear medium, at $\chi = 0$, the amplitude for the Raman field, A_1, is a simple square pulse, of length $\tau_f = \tau_1 + \tau_2$. The amplitude of the Stokes field, A_2, is similar and of the same length, except that there is a sign flip at $\tau = \tau_1$. The initial value condition, at $\tau = 0$, correspond to the medium being initially unexcited.

We shall now solve this problem in the context of the sine–Gordon equation (57). This equation is the integrability condition [23] for the linear system

$$\partial_\chi \psi = U\psi \equiv \begin{pmatrix} -\mathrm{i}\zeta & u_\chi/2 \\ -u_\chi/2 & \mathrm{i}\zeta \end{pmatrix} \psi \tag{59}$$

$$\partial_\tau \psi = V\psi \equiv \lambda \begin{pmatrix} -\cos u & \sin u \\ \sin u & \cos u \end{pmatrix} \psi \qquad \lambda \equiv \mathrm{i}/(4\zeta). \tag{60}$$

Now we wish to apply the inverse scattering method to the region $0 < \tau < \tau_1 + \tau_2$ and $0 < \chi < \chi_f$. The direct scattering problem (59) for $\tau = 0$ and $0 < \chi < \chi_f$ is actually trivial and leads to

$$S(\tau = 0) = \mathbf{1}. \tag{61}$$

Note that the initial scattering data are therefore trivial, and in fact, no solitons or radiation can be initially present. However, as we shall see, the τ-evolution will generate the radiation and the solitons. The radiation and soliton(s) will appear to be 'injected' into the physical region from the boundary at $\chi = 0$.

The τ-evolution of the *effective* S-matrix is governed by

$$\partial_\tau S = -SV(0,\tau) = -\lambda S \times \begin{cases} M_1 & 0 < \tau < \tau_1 \\ M_2 & \tau_1 < \tau < \tau_f \end{cases} \tag{62}$$

$$M_1 \equiv \begin{pmatrix} \cos u_0 & -\sin u_0 \\ -\sin u_0 & -\cos u_0 \end{pmatrix} \qquad M_2 \equiv \begin{pmatrix} \cos u_0 & \sin u_0 \\ \sin u_0 & -\cos u_0 \end{pmatrix} \tag{63}$$

$$M_1^2 = M_2^2 = \mathbf{1}. \tag{64}$$

The solution of (62) at $\tau = \tau_f$ is easily found as

$$S(\tau_f) = (\mathbf{1}\cosh(\lambda\tau_1) - M_1\sinh(\lambda\tau_1))(\mathbf{1}\cosh(\lambda\tau_2) - M_2\sinh(\lambda\tau_2)) \tag{65}$$

and with the notation corresponding to (6), we find the *effective* reflection coefficient c as

$$c = \sin u_0 \frac{\alpha_1 - \alpha_2 - 2C}{(\alpha_1 - C)(\alpha_2 - C) - 1 + C^2} \tag{66}$$

where we have used the abbreviations

$$\alpha_j \equiv \coth(\lambda\tau_j) \qquad C \equiv \cos u_0. \tag{67}$$

Therefore, c, considered as a function of λ (or of ζ), has poles where

$$(\coth(\lambda\tau_1) - C)(\coth(\lambda\tau_2) - C) = 1 - C^2. \tag{68}$$

Let us detail the solution of this equation. First, note that the equation is symmetric in τ_1 and τ_2. Fix the values of τ_1 and τ_2, and consider the left-hand side of (68) as a function of λ, for λ real. As λ varies from 0 to ∞, the left-hand side decreases monotonically from ∞ to $(1-C)^2$, which is less than the right-hand side, of value $1 - C^2$. Thus there is exactly one solution, $\lambda = \lambda_0 > 0$. Consider now the dependence of λ_0 on τ_1 and τ_2. One can easily show that $\lambda_0(\tau_1, \tau_2)$ is a monotonically increasing function of each variable. Consider $\tau_1 > 0$, but fixed, and $\tau_2 \to \infty$. Now λ_0 is seen to reach a finite limit. If we look at what this implies for $\zeta_0 = \mathrm{i}/(4\lambda_0)$, we see that ζ_0 will move from zero, up along the positive imaginary axis, and will approach some finite value.

Now let us discuss this pole condition, equation (68), in various limits.

(a) The condition $\tau_2 \to 0$ corresponds to no sign-flip in the amplitude of A_2. When this is true, the pole condition becomes

$$\coth(\lambda\tau_1) = C \tag{69}$$

or

$$\mathrm{e}^{2\lambda\tau_1} = -\mathrm{e}^{d_0} \qquad d_0 \equiv \log[(1+C)/(1-C)] > 0. \tag{70}$$

There is an infinity of poles along a straight line,

$$\lambda_{\pm k} = \frac{1}{2\tau_1}(d_0 \pm (2k-1)\mathrm{i}\pi) \qquad k = 1, 2, \ldots. \tag{71}$$

This means that when transformed to the ζ-plane, these poles will be along a circle that just touches the real ζ-axis at the origin, and cuts the imaginary axis at

$$\zeta_A = \frac{\mathrm{i}\tau_1}{2d_0}. \tag{72}$$

Furthermore, $\zeta = 0$ will be a cluster point. This situation, with no phase jump, corresponds exactly to the superfluorescence problem with *constant initial tipping angle*, cf [8]. Note that the diameter of the circle is growing proportional to τ_1. The poles are symmetrically placed about the imaginary axis, and there is no pole exactly on the imaginary ζ-axis. All of these poles are in the upper-half ζ-plane. They appear in pairs, $\zeta_{-k} = -\zeta_k^*$, and each such pair corresponds to a breather. These results are in agreement with [24] as well as with [9].

(b) When τ_2 is some multiple of τ_1, i.e. $\tau_2 = n\tau_1 > 0, n = 1, 2, 3, \ldots$, it is easy to find all solutions λ_k from the real solution $\lambda_0 \equiv d_n/2\tau_1$. The positive numbers d_n introduced here are functions of C and n only and are monotonically decreasing with increasing n. d_1, in particular, is determined as the real solution of

$$\coth(d_1/2) = C + \sqrt{1 - C^2}$$

that is

$$d_1 = \log\left[1 + \frac{1}{C}\left(1 + \sqrt{\frac{1+C}{1-C}}\right)\right]. \tag{73}$$

For general n, all solutions are found in the form

$$\lambda_k = \frac{1}{2\tau_1}(d_n + 2k\mathrm{i}\pi) \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{74}$$

The poles in the ζ-plane are again located on the corresponding circle.

(c) The problem simplifies for large τ_2, corresponding to case (b) and $n \gg 1$. When $\tau_2\,\mathrm{Re}(\lambda) \gg 1$ then (68) becomes

$$\coth(\lambda\tau_1) = 1 + 2C$$

or

$$\mathrm{e}^{2\lambda\tau_1} = \mathrm{e}^{d_\infty} \qquad d_\infty \equiv \log(1 + 1/C) \tag{75}$$

which has the solution

$$\lambda_k = \frac{1}{2\tau_1}(d_\infty + 2k\mathrm{i}\pi) \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{76}$$

For the physical cases of interest, we would normally have $A_2 \ll A_1$ at $\chi = 0$. Then we would have $C \simeq 1$, $d_\infty \simeq \log 2 = 0.693$. Consider now the poles $\zeta_k = \mathrm{i}/(4\lambda_k)$. We see that there will be one on the imaginary axis, $\zeta_0 = \mathrm{i}\tau_1/2d_\infty$, and this one will be well separated from all others.

For solving the inverse scattering problem, the residue of $c(\zeta)$ at the poles ζ_k is required. This leads to a rather involved expression in general, but simplifies for the particular cases considered above,

$$\mathrm{Res}(\zeta_k) = -c_k = \frac{\mathrm{i}\tau_1}{8\lambda_k^2 \sin u_0} \times \begin{cases} 2 & \text{(a)} \\ -1 & \text{(b)} \qquad n = 1 \\ \dfrac{-2\exp(-2\tau_2\lambda_k)}{\cos u_0(1 - \cos u_0)} & \text{(c).} \end{cases} \tag{77}$$

It is useful to have in mind some elementary properties of solitons and breathers of the sine–Gordon equation, over the full χ-axis, when one considers these solutions applied to our

SRS system. A single pole at $\zeta = \mathrm{i}\eta_0$ corresponds to a soliton with its spatial width being $\Delta\chi = 1/2\eta_0$ and its location in space being

$$\chi_0 = \eta_0^{-1} \log(2\eta_0/|c_0|). \tag{78}$$

The temporal width of the soliton is $\Delta\tau = 2\eta_0$.

For a breather, corresponding to the pair of poles, $\zeta_k = \xi_k + \mathrm{i}\eta_k$ and $\zeta_{-k} = -\xi_k + \mathrm{i}\eta_k$, with $c_{-k} = -c_k^*$, one obtains a spatial width $\Delta\chi = 1/2\eta_k$, and a temporal width $\Delta\tau = 2(\xi_k^2 + \eta_k^2)/\eta_k$. The location in space is given by the same formula (78), but with subscripts 0 replaced by k. The maximum amplitudes are found as

$$|X|_{\max} = 4\eta_k \qquad |A_1|_{\max} = \frac{2\xi_k\eta_k}{\xi_k^2 + \eta_k^2}. \tag{79}$$

Now let us return to the finite interval $0 < \chi < \chi_f$ or, maybe, to the half-line $0 < \chi$. For u_0 sufficiently small, in cases (a) and (b), we find the breathers with low k located in the unphysical region, $\chi < 0$. Thus these breathers would never be observed, except for their tails, at best. On the other hand, for large k, they become very broad in space and very low in amplitude. Thus these would only contribute some long-wavelength structure to the background. So there is almost no chance that any approximation, with a finite number of poles, could lead to an adequate description. Instead, in these cases, one should look for some other approximation. Indeed, the corresponding superfluorescence problem equivalent to our case (a) was solved by the saddle-point method [8].

For case (c), the situation is completely different. Here we have a single soliton, which is well separated from a series of breathers. Proceeding, we obtain

$$c_0 = \frac{2\mathrm{i}\tau_1^3}{d_\infty^2 u_0^3} \mathrm{e}^{-d_\infty \tau_2/\tau_1} \tag{80}$$

$$|c_0/c_1| = 1 + (2\pi/d_\infty)^2 \simeq 58. \tag{81}$$

Thus for τ_2 sufficiently large we obtain $\chi_0 > 0$ as the location of the single soliton and this is well separated from the nearest breather. And we find maximum values

$$|X^{\mathrm{sol}}|_{\max} \simeq 5.8\tau_1 \qquad |A_1^{\mathrm{sol}}|_{\max} \simeq 1 \tag{82}$$

for the soliton and

$$|X^{\mathrm{bre}}|_{\max} \simeq 1.43\tau_1 \qquad |A_1^{\mathrm{bre}}|_{\max} \simeq 0.22 \tag{83}$$

for the first breather.

It might be illustrative to look at an example where we have a region of a pure Stokes pulse, following after the laser pulse. Let us take $\tau > \tau_0 \geqslant \tau_1$. Then the evolution of the *effective* reflection coefficient is simply

$$c(\tau) = c(\tau_0)\,\mathrm{e}^{-2\lambda(\tau-\tau_0)}. \tag{84}$$

In this region, the motion of the poles has ceased. This is a stable arrangement: no laser pulse and only pure Stokes. We note that due to the exponential factor, the single soliton—when there is a pole on the imaginary axis—and the breathers, will always reach the physical region, $\chi > 0$, for some sufficiently large τ. And because of their differing velocities, the soliton and the individual breathers become more and more well separated, as τ increases.

5. Summary and conclusion

We developed the IST method for solving a Goursat problem with a finite x-interval. The essential restriction to be made is that the linear equation governing the temporal evolution of scattering data, equation (7), must contain only negative powers of the spectral parameter. That is, the method is applicable to the sine–Gordon equation, self-induced transparency (SIT), stimulated Raman scattering and many other similar problems, but not, however, to Korteweg–de Vries and nonlinear Schrödinger equations, and other equations that do not have characteristics. The recipe following from our analysis is extremely simple: establish the equation for the temporal development of the S-matrix in the finite interval and simply omit the terms containing the data from the right-hand border. The *effective* S-matrix resulting in this way fully describes the physics in the physical region.

By means of two examples—SHG and SRS—we demonstrated that this method does work for AKNS problems with $r = q^*$ as well as for those with $r = -q^*$ reductions. As a somewhat more involved problem, the propagation of an optical pulse in a two-photon absorber will be treated elsewhere by the same method [25].

Finally, we wish to add some remarks concerning SRS solitons. Our present study in section 4 shows how solitons will evolve in the absence of perturbations. However, this cannot yield a realistic description of the experiment by Drühl *et al* [22], simply because damping has been completely neglected. In the actual experiment, the damping times were of the order of the soliton width. However, it has been known for a long time, that a perturbative analysis of this IST [26] has given rather good agreement with the experiment.

On the other hand, one can find in the literature the statement by Claude *et al* [27] that '... the spike of pump radiation' found in the experiment [22] 'is not a soliton, but merely a special manifestation of the *continuous spectrum*...'. This conclusion has been based on some 'general theorem', according to which, no discrete spectrum can appear when initially there is none. This is certainly true for the full-line problem. However, as illustrated herein, it is clearly false for a finite or semi-infinite interval problem.

Our intention in this present treatment of SRS is simply to allow others to better understand the nature of SRS transient solitons. Principally, the effective S-matrix formalism presented here is not necessary for understanding their nature, but it is helpful. In [28], the mathematical equivalence of SRS and sharp-line SIT solitons was pointed out in some detail, and, in particular, it was there discussed how solitons can enter or leave a finite x–t region. Lastly, we make the following simple observation: at some level of idealization, the SRS solitons observed by Drühl *et al* will be described by the sine–Gordon equation. It is well known that the sine–Gordon equation does not have 'spikes', it has solitons.

Acknowledgments

HS wishes to acknowledge financial support by the Max Planck Society. The research of DJK was supported in part by the US Air Force Office of Scientific Research.

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